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This article corrects two errors that appear in the study by Banerjee and Bhattacharyya (1976). It shows that the mixture of inverse Gaussian distributions can look like a better model of purchase frequencies than implied by Banerjee and Bhattacharyya’s work.

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Banerjee and Bhattacharyya (1976), from here on BB, use the method of moments to calibrate a compound inverse Gaussian (IG) model. We correct two errors in their article that affect the estimated model parameters and the predictions. The expressions for $E(\psi)$ and $E(\psi \lambda)$ should read (see the App.)

$$E(\psi) = \frac{1}{\beta} + \frac{\alpha}{\beta(\gamma - 3)} H_{\gamma - 1}(\xi), \quad \gamma > 3,$$

(A.3)

and

$$E(\psi \lambda) = \frac{\gamma - 1}{\alpha \beta \gamma} H_{\gamma - 1}(\xi) \left( \frac{\gamma + 1}{\alpha \beta} \right),$$

(A.4)

where $\xi = [(\gamma - 1)/(\alpha \beta)]^{1/2}$, $\xi_1 = [(\gamma - 3)/(\alpha \beta)]^{1/2}$, and $\xi_2 = [(\gamma + 1)/(\alpha \beta)]^{1/2}$. The system of equations is such that:

$$E(\psi) = .653,$$

$$E(\psi \lambda) = 1.095,$$

$$E(\lambda) = 2.046.$$

With the corrected equations, model calibration with nonlinear least squares yields $\hat{\alpha} = .357$, $\hat{\beta} = 1.973$, and $\hat{\gamma} = 3.047$. To verify the correctness of (A.3) and (A.4), we computed the two means by numerical integration (over the marginal density of $\psi$ and over the joint density of $\psi$ and $\lambda$) with the new parameter estimates and found .653 and 1.094, respectively. When we plug the estimates obtained by BB into (A.3) and (A.4), we find .639 and 1.132, respectively, which represents $-2.1$ and $+3.5$ percentage errors. Interestingly, when we plug BB’s estimates into their own equation for $E(\psi)$ and for $E(\psi \lambda)$, we find .56 and .607, respectively. Due to the difference with 1.095, it is likely that there is a typographical error in BB’s equation for $E(\psi \lambda)$. The predictions for the proportion of nonbuyers and for the tails of the purchase frequencies as a function of the length of the time period are as tabulated in Table 1.

An apparently minor error in $E(\psi)$, the weekly mean purchase frequency, turns into a relatively substantial error in the proportion of nonbuyers as the length of the observation period becomes large (i.e., 4 to 5 months). BB’s predictions underestimate the proportion of nonbuyers. The compound IG exhibits somewhat fatter tails than predicted by BB. We can conjecture that, as the length of the observation period becomes large, the compound IG model may look like a better model of purchase frequency than implied by BB due to an improved description of heterogeneity. Maybe, this article can rehabilitate a model that appears to have lost ground to the negative binomial distribution (NBD) in the modeling of purchasing frequencies.

**APPENDIX: EXPECTED VALUES OF $\psi$ AND $\psi \lambda$**

This appendix sketches the derivation of the expected values of $\psi$ and $\psi \lambda$. A key intermediate expression for later use is the result for integrals of the type $\int_0^\infty x[a(x - b)^2 + 1]^{-c} dx$, which is derived in the following. The antiderivative of the integrand results in

$$\int x[a(x - b)^2 + 1]^{-c} dx$$

$$= \frac{(a[b - x]^2 + 1)^c - a[b(x - b)^2 + 1]}{2a(c - 1)}$$

$$- b[b - x] F_1 \left( \frac{1}{2}, \frac{3}{2}; -a(b - x)^2 \right),$$

(A.1)

where $F_1(\cdot)$ is the Gauss hypergeometric function. We evaluate (A.1) at 0 and $\infty$:

- $x$ equals 0
  - When $x$ is equal to 0, (A.1) becomes
    $$\frac{[a(b^2 + 1)^c - ab^2 - 1]}{2a^{(c - 1)}} - b^2 F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}; -ab^2 \right).$$

- $x$ approaches $\infty$
  - When $x$ approaches $\infty$, the first part of (A.1) becomes $1/[2a(c - 1)]$ as $x$ approaches $\infty$. Further, we know that (Abramowitz and Stegun 1972, equation 15.3.4)
    $$b[b - x] F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}; -a(b - x)^2 \right)$$
    $$= \frac{b[b - x]}{a[b(x - b)^2 + 1]} F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}; \frac{a(b - x)^2}{a(b - x)^2 + 1} \right).$$

When $x$ approaches $\infty$, the latter equation results in

$$- \frac{b}{\sqrt{a^2}} F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}; 1 \right) = - \frac{b}{2} \left( \sqrt{\frac{a}{\Gamma(c - 1/2)}} \right).$$

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Combining and rearranging these results, we obtain:

\[
\int_0^\infty x[a(x - b)^2 + 1]^{-c} dx = \frac{(ab^2 + 1)^{1-c}}{2c} + \frac{b}{2a} \frac{\Gamma(c-1/2)}{\Gamma(c)} + \frac{b^2}{2} F_1 \left( \frac{1}{2}, \frac{3}{2}, -ab^2 \right).
\]

To find \(E(\psi)\), we evaluate \(\int_0^\infty \psi \cdot p(\psi) d\psi\), where \(p(\psi)\) is the marginal density of \(\psi\) [BB, (2.3)]:

\[
\int_0^\infty \psi \cdot p(\psi) d\psi = \left( \frac{\beta}{\alpha} \right)^{1/2} H_{1/2}(\psi) \left( \frac{\nu}{2} \right)^{-1/2} \int_0^\infty \left[ 1 + \frac{\beta}{\alpha} (\psi - \frac{1}{\beta}) \right]^{-\nu/2} d\psi,
\]

where \(\nu = \gamma - 1, \xi = [(\gamma - 1)/(\alpha \beta)]^{1/2}\), \(B(\cdot)\) is the beta function, and \(H_{1/2}(\cdot)\) is the cumulative distribution function of the \(t\) distribution with \(\nu\) degrees of freedom. With \(a = \beta/\alpha, b = 1/\beta\), and \(c = \gamma/2\), the integral in this expression can be solved using (A.2). Taking into account the hypergeometric function representation of the \(t\) distribution (Amos 1964), we find that:

\[
E(\psi) = \left( \frac{\beta}{\alpha} \right)^{1/2} \left[ H_{1/2}(\psi) \left( \frac{\nu}{2} \right)^{-1/2} \right]^{-1} \times \left[ \frac{\alpha^{1/2}}{\beta^{1/2}} B \left( \frac{\nu}{2}, \frac{1}{2} \right) H_{1/2}(\psi) + \frac{1 + 1/(\alpha \beta)}{\beta^2 (\nu - 2)} \right].
\]

Letting \(\xi_1 = [(\gamma - 3)/(\alpha \beta)]^{1/2}\) and with the probability density function of the \(t\) distribution with \(\nu\) degrees of freedom, \(h_\nu(\cdot)\), we find the expected value of \(\psi\) to be:

\[
E(\psi) = \frac{1}{\beta} + \frac{\alpha}{\beta (\nu - 3) H_{\nu-3}(\xi_1)} \quad \gamma > 3.
\]

The expected value of \(\psi\) results by evaluating \(\int_0^\infty \psi \cdot p(\psi, \lambda) d\psi d\lambda\), where \(p(\psi, \lambda)\) is the joint density of \(\psi\) and \(\lambda\) [BB, (2.2)]. Solving the integral over \(\lambda\) yields:

\[
E(\psi, \lambda) = E(\psi | \lambda) = \left( \frac{\beta}{\alpha} \right)^{1/2} \left[ H_{1/2}(\psi) \left( \frac{\nu}{2} \right)^{-1/2} \right]^{-1} \times \int_0^\infty \psi \left[ 1 + \frac{\beta}{\alpha} (\psi - \frac{1}{\beta}) \right]^{-\nu/2 - 1/2} d\psi.
\]

The remaining integral over \(\psi\) may be rearranged using (A.2), where \(a = \beta/\alpha, b = 1/\beta\), and \(c = \gamma/2\). Defining \(\xi_2 = (\gamma + 1)/(\alpha \beta)^{1/2}\), we find that:

\[
E(\psi, \lambda) = \left( \frac{\beta}{\alpha} \right)^{1/2} \left[ H_{1/2}(\psi) \left( \frac{\nu}{2} \right)^{-1/2} \right]^{-1} \times \left[ \frac{\alpha^{1/2}}{\beta^{1/2}} B \left( \frac{\nu}{2}, \frac{1}{2} \right) H_{1/2}(\psi) + \frac{1 + 1/(\alpha \beta)}{\beta^2 (\nu - 2)} \right].
\]

Simplifying this expression yields the following result:

\[
E(\psi, \lambda) = \frac{1}{\alpha \beta} \left[ H_{\nu-1}(\xi_2) \left( 1 + \alpha \beta \right) \right] \gamma \frac{\nu + 1}{\alpha \beta} \frac{H_{\nu+1}(\xi_2)}{H_{\nu+1}(\xi_1)}
\]

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